

A unified formulation of Gaussian vs. sparse stochastic processes— Part I: Continuous-domain theory

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Abstract

We introduce a general distributional framework that results in a unifying description and characterization of a rich variety of continuous-time stochastic processes. The cornerstone of our approach is an innovation model that is driven by some generalized white noise process, which may be Gaussian or not (e.g., Laplace, impulsive Poisson or alpha stable). This allows for a conceptual decoupling between the correlation properties of the process, which are imposed by the whitening operator L , and its sparsity pattern which is determined by the type of noise excitation. The latter is fully specified by a Lévy measure. We show that the range of admissible innovation behavior varies between the purely Gaussian and super-sparse extremes. We prove that the corresponding generalized stochastic processes are well-defined mathematically provided that the (adjoint) inverse of the whitening operator satisfies some L_p bound for $p \geq 1$. We present a novel operator-based method that yields an explicit characterization of all Lévy-driven processes that are solutions of constant-coefficient stochastic differential equations. When the underlying system is stable, we recover the family of stationary CARMA processes, including the Gaussian ones. The approach remains valid when the system is unstable and leads to the identification of potentially useful generalizations of the Lévy processes, which are sparse and non-stationary. Finally, we show how we can apply finite difference operators to obtain a stationary characterization of these processes that is maximally decoupled and stable, irrespective of the location of the poles in the complex plane.

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I. INTRODUCTION

In recent years, the research focus in signal processing has shifted away from the classical linear paradigm, which is intimately linked with the theory of stationary Gaussian processes [1], [2]. Instead of considering Fourier transforms and performing quadratic optimization, researchers are presently favoring wavelet-like representations and have adopted sparsity as design paradigm [3]–[7]. The property that a signal admits a sparse expansion can be exploited elegantly for compressive sensing, which is presently a very active area of research (cf. special issue of the Proceedings of the IEEE [8], [9]). The concept is equally helpful for solving inverse problems and has resulted in significant algorithmic advances for the efficient resolution of large scale ℓ_1 -norm minimization problems [10]–[12].

The current formulations of compressed sensing and sparse signal recovery are fundamentally deterministic. By drawing on the analogy with the classical theory of signal processing, it is likely that further progress may be achieved by adopting a statistical (or estimation theoretic) point of view. This stands as our primary motivation for the investigation of the present class of continuous-time stochastic processes, the greater part of which is sparse by construction. These processes are specified as a superset of the Gaussian ones, which is essential for maintaining backward compatibility with traditional statistical signal processing.

The inspiration for this work is provided by the innovation approach to system modeling—a standard technique in statistics and control theory that is well developed in the discrete setting and often favored by engineers. Innovation models are also used in signal processing for the investigation of continuous-time stationary Gaussian stochastic processes [1], [13]. Non-Gaussian variants of such models are easy to set up in the discrete world, but they do result in harder identification problems [14]–[16]. An active topic of research is the determination of a proper noise input to simulate signals with prescribed marginal distributions [17], [18]. By contrast, there is comparatively little work on continuous-domain innovations for the specification of non-Gaussian or/and non-stationary processes due to the inherent difficulty of rigorously defining non-Gaussian white noise in the continuous domain. The proper mathematical framework exists and was developed by the Russian school of mathematics in the 1960s [19], but has hardly been used by practitioners until now. This is mainly due to the widespread acceptance of stochastic integration (Itô calculus) in the advanced theory of stochastic processes [20]–[23], which avoids the direct handling of white noise and tempered distributions.

By following up on our initial work on the generation of piecewise-smooth signals from random streams of Dirac impulses (Poisson white noise) [24], our present aim is to set the foundations of a

comprehensive theory of continuous-domain stochastic processes based on the simple, unifying principle of the filtering of special brands of (non-Gaussian) white noise. While the concept remains applicable in multiple dimensions, we focus on the time domain (1-D signals), and provide a systematic treatment of systems that are described by ordinary differential equations, including some novel twists for the non-stable scenarios, which opens the door to interesting generalizations. The primary contributions are:

- 1) The extension of our prior innovation models to the broadest possible class of white noises beyond the Gaussian and impulsive Poisson categories: We show that each brand is uniquely specified by a Lévy measure that conditions the degree of sparsity of the process. The Gaussian processes are the least sparse ones; the Poisson processes are intermediate with their level of sparsity being controlled by the rate parameter λ [24]. The sparsest processes are the alpha-stable ones whose marginal distributions are heavy tailed with unbounded variance [22], [25].
- 2) The systematic investigation of processes that are ruled by constant-coefficient SDEs together with the proposal of a generic operator-based method of solution: When the underlying system is stable, we recover the complete family of (non-Gaussian) continuous-time autoregressive moving average (CARMA) processes (see also the work of Brockwell for an equivalent state-space characterization that relies on stochastic integrals [26]). The further reaching aspect of our formulation is that the method remains applicable in the non-stable case and that it leads to some interesting generalizations of Lévy processes, which are non-stationary.
- 3) The generalization/extension of our previous stability and existence results (cf. [24, Theorem 2], [27, Theorem 1.3]) for our enlarged class of stochastic processes: In essence, we are replacing the basic L_2 -boundedness requirement that is central to the continuous-time Gaussian theory by a more robust L_p condition (cf. Theorem 3); the case $p = 1$ is required for the non-symmetric Poisson processes, while the range of values $p \in (0, 2)$ becomes appropriate for the alpha stable processes.

The paper is organized as follows. In Section II, we briefly review the foundations of Gelfand's theory of generalized stochastic processes. In particular, we characterize the complete class of admissible continuous-time white noise processes and give some argumentation as to why the non-Gaussian brands are inherently sparse. We then introduce our general innovation model in Section III, while providing a novel operator-based method for the solution of SDE. In Section IV, we make use of Gelfand's formalism to fully characterize our extended class of (non-Gaussian) stochastic processes including the special cases of CARMA and N th-order generalized Lévy processes. We also introduce the notion of generalized increment process and some corresponding B-spline calculus, which allows for a common stationary

treatment of the two latter classes of processes, irrespective of any stability consideration. Finally, in Section V, we take a non-conventional look at the classical Lévy processes [23], [28], [29], which correspond to the simplest (unstable) instance of our extended formulation (single pole at the origin). For higher-order illustrations of sparse processes, we refer to our companion paper [30], which is more specifically devoted to the study of the discrete-time implication of the theory. The notation, which is common to both papers, is summarized in [30, Table I].

II. PRELIMINARIES

The purpose of this section is to introduce the distributional formalism that is required for the proper definition of continuous-time white noise. We start with a brief summary of some required notions in functional analysis, which also serves us to set the notation. We then introduce the fundamental concept of characteristic functional which constitutes the foundation of Gelfand's theory of generalized stochastic processes. We proceed by giving the complete characterization of the possible types of continuous-domain white noises—not necessarily Gaussian—which will be used as universal input for our innovation models. We conclude the section by showing that the non-Gaussian brands of noises that are allowed by Gelfand's formulation are intrinsically sparse, a property that has not been emphasized before (to the best of our knowledge).

A. Functional and distributional context

The L_p -norm of a function $f(t)$ is $\|f\|_p = \left(\int_{\mathbb{R}} |f(t)|^p dt\right)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|f\|_{\infty} = \text{ess sup}_{t \in \mathbb{R}} |f(t)|$ for $p = +\infty$ with the corresponding Lebesgue space being denoted by L_p . The concept is extendable for characterizing the rate of decay of functions. To that end, we introduce the weighted $L_{p,\alpha}$ spaces with $\alpha \in \mathbb{R}^+$

$$L_{p,\alpha} = \{f(t) \in L_p : \|f\|_{p,\alpha} < +\infty\}$$

where the α -weighted L_p -norm of f is defined as

$$\|f\|_{p,\alpha} = \|(1 + |t|^\alpha)f(t)\|_p.$$

Hence, the statement $f \in L_{\infty,\alpha}$ implies that $f(t)$ is decaying at least like $(1/|t|^\alpha)$ as t tends to $\pm\infty$; more precisely, that $|f(t)| \leq \frac{\|f\|_{\infty,\alpha}}{1+|t|^\alpha}$ almost everywhere. In particular, this allows us to infer that $L_{\infty,\frac{1}{p}+\epsilon} \subset L_p$ for any $\epsilon > 0$ and $p \geq 1$. Another obvious inclusion is $L_{p,\alpha} \subseteq L_{p,\alpha_0}$ for any $\alpha \geq \alpha_0$. In the limit, we end up with the space of rapidly-decreasing functions $\mathcal{R} = \{f(t) : \|f\|_{\infty,m} < +\infty, \forall m \in \mathbb{Z}^+\}$, which is included in all the others.

We use $\varphi(t)$ to denote a generic function in Schwartz's class \mathcal{S} of rapidly-decaying and infinitely-differentiable test functions. Specifically, Schwartz's space is defined as:

$$\mathcal{S} = \left\{ \varphi(t) \in C^\infty : \|D^n \varphi\|_{\infty, m} < +\infty, \forall m, n \in \mathbb{Z}^+ \right\},$$

with the operator notation $D^n = \frac{d^n}{dt^n}$ and the convention that $D^0 = \text{Id}$ (identity). \mathcal{S} is a complete topological vector space. Its topological dual is the space of tempered distributions \mathcal{S}' ; a distribution $\phi \in \mathcal{S}'$ is a continuous linear functional on \mathcal{S} that is characterized by a duality product rule $\phi(\varphi) = \langle \phi, \varphi \rangle = \int_{\mathbb{R}} \phi(t) \varphi(t) dt$ with $\varphi \in \mathcal{S}$ where the right-hand side expression has a literal interpretation as an integral only when $\phi(t)$ is true function of t . The prototypical example of a tempered distribution is the Dirac distribution δ , which is defined as $\delta(\varphi) = \langle \delta, \varphi \rangle = \varphi(0)$. In the sequel, we will drop the explicit dependence of the distribution on the generic test function $\varphi \in \mathcal{S}$ and simply write ϕ or even $\phi(t)$ (with an abuse of notation).

Let T be a continuous¹ linear operator that maps \mathcal{S} into itself (or eventually some enlarged topological space such as L_p). It is then possible to extend the action of T over \mathcal{S}' (or an appropriate subset of it) based on the definition $\langle T\phi, \varphi \rangle = \langle \phi, T^*\varphi \rangle$ if T^* is the adjoint of T which maps φ to another test function $T^*\varphi \in \mathcal{S}$ continuously. An important example is the Fourier transform whose classical definition is $\mathcal{F}\{f\}(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-j\omega t} dt$. Since \mathcal{F} is a self-adjoint \mathcal{S} -continuous operator, it is extendable to \mathcal{S}' based on the adjoint relation $\langle \mathcal{F}\phi, \varphi \rangle = \langle \phi, \mathcal{F}\varphi \rangle$ for all $\varphi \in \mathcal{S}$ (generalized Fourier transform).

A linear, shift-invariant (LSI) operator that is well-defined over \mathcal{S} can always be written as a convolution product:

$$T_{\text{LSI}}\varphi(t) = (h * \varphi)(t) = \int_{\mathbb{R}} h(\tau) \varphi(t - \tau) d\tau$$

where $h(t) = T_{\text{LSI}}\delta(t)$ is the impulse response of the system. The adjoint operator is the convolution with the time-reversed version of h :

$$h^\vee(t) = h(-t).$$

The better-known categories of LSI operators are the BIBO-stable (bounded input, bounded output) filters, and the ordinary differential operators. While the latter are not BIBO-stable, they do work well with test functions.

¹An operator T is a continuous map from a separable topological vector space \mathcal{V} into another one iff. $\varphi_k \rightarrow \varphi$ in the topology of \mathcal{V} implies that $T\varphi_k \rightarrow T\varphi$ in the topology (or norm) of the second space. If the two spaces coincide, we say that T is \mathcal{V} -continuous.

1) *L_p -stable LSI operators:* The BIBO-stable filters correspond to the case where $h \in L_1$, or, more generally, when h corresponds to a complex-valued Borel measure of bounded variation. The latter extension allows for discrete filters of the form $h_d(t) = \sum_{n \in \mathbb{Z}} d[n] \delta(t - n)$ with $d[n] \in \ell_1$. We will refer to these filters as L_p -stable because they are bounded in all L_p -norms (by Young's inequality). L_p -stable convolution operators satisfy the properties of commutativity, associativity, and distributivity with respect to addition.

2) *\mathcal{S} -continuous LSI operators:* For an L_p -stable filter to yield a Schwartz function as output, it is necessary that its impulse response (continuous or discrete) be rapidly-decaying. In fact, the condition $h \in \mathcal{R}$ (which is much stronger than integrability) ensures that the filter is \mathcal{S} -continuous. The n th-order derivative D^n and its adjoint $D^{n*} = (-1)^n D^n$ are in the same category. The n th-order *weak* derivative of the tempered distribution ϕ is defined as $D^n \phi(\varphi) = \langle D^n \phi, \varphi \rangle = \langle \phi, D^{n*} \varphi \rangle$ for any $\varphi \in \mathcal{S}$. The latter operator—or, by extension, any polynomial of distributional derivatives $P_N(D) = \sum_{n=1}^N a_n D^n$ with constant coefficients $a_n \in \mathbb{C}$ —maps \mathcal{S}' into itself. The class of these differential operators enjoys the same properties as its classical counterpart: shift-invariance, commutativity, associativity and distributivity.

B. Notion of generalized stochastic process

The leading idea in distribution theory is that a generalized function ϕ is not defined through its point values $\phi(t), t \in \mathbb{R}$, but rather through its scalar products $\phi(\varphi) = \langle \phi, \varphi \rangle$ with all “test” functions $\varphi \in \mathcal{S}$. In an analogous fashion, Gelfand defines a generalized stochastic process s via the probability law of its scalar products with arbitrary test functions $\varphi \in \mathcal{S}$, rather than by considering the probability law of its pointwise samples $\{\dots, s(t_1), s(t_2), \dots, s(t_N), \dots\}$, as is customary in the conventional formulation.

Let s be such a generalized process. We first observe that the scalar product $x_1 = \langle s, \varphi_1 \rangle$ with a given test function φ_1 is a conventional (scalar) random variable that is characterized by a probability density $P_1(dx_1)$; the latter is in one-to-one correspondence (via the Fourier transform) with the characteristic function $\hat{p}_1(\omega_1) = \mathcal{E}\{e^{j\omega_1 x_1}\} = \int_{\mathbb{R}} e^{j\omega_1 x_1} P_1(dx_1) = \mathcal{E}\{e^{j\langle s, \omega_1 \varphi_1 \rangle}\}$ where $\mathcal{E}\{\cdot\}$ is the expectation operator. The same applies for the probability density $P_{1,2}(dx_1 dx_2)$ associated with a pair of test functions φ_1 and φ_2 which is the inverse Fourier transform of the 2-D characteristic function $\hat{p}_{1,2}(\omega_1, \omega_2) = \mathcal{E}\{e^{j\langle s, \omega_1 \varphi_1 + \omega_2 \varphi_2 \rangle}\}$, and so forth if one wants to specify higher-order dependencies.

The foundation for the theory of generalized stochastic processes is that one can deduce the complete statistical information about the process from the knowledge of its characteristic form

$$\mathcal{Z}_s(\varphi) = \mathcal{E}\{e^{j\langle s, \varphi \rangle}\} \quad (1)$$

which is a continuous, positive-definite functional over \mathcal{S} such that $\mathcal{Z}_s(0) = 1$. Since the variable φ in $\mathcal{Z}_s(\varphi)$ is completely generic, it provides the equivalent of an infinite-dimensional generalization of the characteristic function. Indeed, any finite dimensional version can be recovered by direct substitution of $\varphi = \omega_1\varphi_1 + \cdots + \omega_N\varphi_N$ in $\mathcal{Z}_s(\varphi)$ where the φ_n are fixed and where $\omega = (\omega_1, \dots, \omega_N)$ takes the role of the N -dimensional Fourier variable. In fact, Gelfand's theory rests upon the principle that specifying an admissible functional $\mathcal{Z}_s(\varphi)$ is equivalent to defining the underlying generalized stochastic process (Bochner-Minlos theorem). The precise statement of this result, which relies upon the fundamental notion of positive-definiteness, is given in Appendix I.

C. White noise processes

We define white noise as a generalized random process that is stationary and whose measurements for non-overlapping test functions are independent. A remarkable aspect of the theory of generalized stochastic processes is that it is possible to deduce the complete class of such noises based on functional considerations only [19]. To that end, Gelfand and Vilenkin consider the generic class of functionals of the form

$$\mathcal{Z}_w(\varphi) = \exp \left(\int_{\mathbb{R}} f(\varphi(t)) dt \right) \quad (2)$$

where $f(u)$ is a continuous function of the scalar variable $u \in \mathbb{R}$. They first argue that this functional specifies an independent noise process iff. \mathcal{Z}_w is positive-definite (as required by the Bochner-Minlos theorem) and $\mathcal{Z}_w(\varphi_1 + \varphi_2) = \mathcal{Z}_w(\varphi_1)\mathcal{Z}_w(\varphi_2)$ whenever φ_1 and φ_2 have non-overlapping support (i.e., $\varphi_1(t) \cdot \varphi_2(t) = 0$). The latter is equivalent to the requirement that $f(u)$ in (2) is such that $f(0) = 0$. They then go on to prove that the complete class of functionals of the form (2) with the required mathematical properties (continuity, positive-definiteness and factorizability) is obtained when

$$f(u) = b_0 + jb'_1 u - \frac{b_2 u^2}{2} + \int_{\mathbb{R} \setminus \{0\}} [e^{jau} - r(a)(1 + jau)] R(da)$$

where R is an other arbitrary positive measure on \mathbb{R} such that $\int_{\mathbb{R} \setminus \{0\}} \min(1, a^2) R(da) < \infty$ and where $r(a)$ is a function such that $r(a) - 1$ has a third-order zero at $a = 0$ and $\int_{|a|>0} [1 - r(a)] R(da) + b_0 = 0$ (to ensure that $f(0) = 0$). In doing so, they actually establish a one-to-one correspondence between the characteristic functional of an independent noise processes of the form (2) and the (classical) characteristic function $e^{f(\omega)} = \mathcal{E}\{e^{j\omega x}\}$ of an infinite divisible scalar random variable x [31], [32]. An equivalent, more concise specification of the complete family of admissible functions $f(u)$ is provided by the Lévy-Khintchine formula

$$f(u) = jb''_1 u - \frac{b_2 u^2}{2} + \int_{\mathbb{R} \setminus \{0\}} [e^{jau} - 1 - jau1_{|a|<1}(a)] V(da) \quad (3)$$

where $1_{|a|<1}(a)$ is the indicator function that takes the value 1 if $|a| < 1$ and zero otherwise. Here V is a Borel measure on $\mathbb{R} \setminus \{0\}$ such that

$$\int_{\mathbb{R} \setminus \{0\}} \min(1, a^2) V(da) < \infty. \quad (4)$$

We will see that the density $V(da)$ actually corresponds to the so-called Lévy measure that enters the definition of Lévy processes [23], [29], [32]. To further our mathematical understanding of the Lévy-Khintchine formula (3), we note that $e^{ja u} - 1 - ja u 1_{|a|<1}(a) = -\frac{1}{2}a^2 u^2 + O(|a u|^3)$ as $a \rightarrow 0$. This ensures that the integral is convergent even when $V(da)$ exhibits a singularity at the origin to the extent allowed by the admissibility condition (4). If the Lévy measure is finite (i.e., $\int_{\mathbb{R}} V(da) < \infty$) or symmetrical ($V(E) = V(-E)$ for any $E \subset \mathbb{R}$), it is then also possible to use the equivalent, simplified form of Lévy function

$$f(u) = j b_1 u - \frac{b_2 u^2}{2} + \int_{\mathbb{R}} (e^{ja u} - 1) V(da)$$

with $b_1 = b_1'' - \int_{a<1} a V(da)$. Concretely, this means that a particular brand of such independent noise process is completely characterized by a triplet $(b_1, b_2, V(da))$. With this latter convention, the Lévy functions and characteristic forms of the three primary types of white noise encountered in the signal processing literature are:

- 1) Gaussian: $b_1 = 0, b_2 = 1, V = 0$

$$\begin{aligned} f_{\text{Gauss}}(u) &= -\frac{|u|^2}{2}, \\ \mathcal{Z}_w(\varphi) &= e^{-\frac{1}{2}\|\varphi\|_{L_2}^2}. \end{aligned} \quad (5)$$

- 2) Compound Poisson: $b_1 = 0, b_2 = 0, V(da) = \lambda p_a(a) da$ with $\int_{\mathbb{R}} p_a(a) da = \hat{p}_a(0) = 1$,

$$\begin{aligned} f_{\text{Poisson}}(u; \lambda, p_a) &= \lambda \int_{\mathbb{R}} (e^{ja u} - 1) p_a(a) da, \\ \mathcal{Z}_w(\varphi) &= \exp \left(\lambda \int_{\mathbb{R}} \int_{\mathbb{R}} (e^{ja \varphi(t)} - 1) p_a(a) da dt \right). \end{aligned} \quad (6)$$

- 3) Symmetric alpha-stable (S α S): $b_1 = 0, b_2 = 0, V(da) = \frac{C_\alpha}{|a|^{\alpha+1}} da$ with $0 < \alpha < 2$ and $C_\alpha = \frac{\sin(\frac{\pi \alpha}{2})}{\pi}$ a suitable normalization constant,

$$\begin{aligned} f_\alpha(u) &= \frac{-|u|^\alpha}{\alpha!}, \\ \mathcal{Z}_w(\varphi) &= e^{-\frac{1}{\alpha!} \|\varphi\|_{L_\alpha}^\alpha}. \end{aligned} \quad (7)$$

The latter follows from the fact that $\frac{-|\omega|^\alpha}{\alpha!}$ is the generalized Fourier transform of $\frac{C_\alpha}{|t|^{\alpha+1}}$ with the convention that $\alpha! = \Gamma(\alpha + 1)$ where Γ is Euler's Gamma function [33].

While none of these noises has a classical interpretation as a random function of t , we can at least provide an explicit description of the Poisson noise as a random sequence of Dirac impulses (cf. [24, Theorem 1])

$$w_\lambda(t) = \sum_k a_k \delta(t - t_k)$$

where the t_k are random locations that are uniformly distributed over \mathbb{R} with density λ , and where the weights a_k are i.i.d. random variables with probability density function (PDF) $p_a(a)$.

D. Gaussian versus sparse categorization

To get a better understanding of the underlying class of objects, we propose to probe them through some localized analysis window φ , which will yield a conventional i.i.d. random variable $x = \langle w, \varphi \rangle$ with some probability density function (PDF) $p_\varphi(x)$. The most convenient choice is to pick the rectangular analysis window $\varphi(t) = \text{rect}(t) = 1_{[-\frac{1}{2}, \frac{1}{2}]}(t)$. By using the fact that $e^{j\omega \text{rect}(t)} - 1 = e^{j\omega} - 1$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$, and zero otherwise, we find that the characteristic function of x is simply given by

$$\hat{p}_{\text{rect}}(\omega) = \mathcal{Z}_w(\omega \cdot \text{rect}(t)) = \exp(f(\omega)),$$

which corresponds to the generic (Lévy-Khinchine) form associated with an infinitely-divisible distribution [29], [32], [34]. The above result makes the mapping between generalized white noise processes and classical infinite-divisible (id) laws² explicit: The “canonical” id distribution of w , $p_{\text{id}}(x) = p_{\text{rect}}(x)$, is obtained by observing the noise through a rectangular window. Conversely, given the logarithm of the characteristic function of an id distribution, $\log(\mathcal{F}\{p_{\text{id}}(x)\}(\omega)) = f(\omega)$, we can specify a corresponding generalized white noise process w via the characteristic form $\mathcal{Z}_w(\varphi)$ by merely substituting the frequency variable ω by the generic test function $\varphi(t)$, adding an integration over \mathbb{R} and taking the exponential as in (2).

We will now argue that this class of models allows for a range of behaviors that varies between the purely Gaussian and sparse extremes. In the context of Lévy processes, these are often referred to as the diffusive and jump modes. To make our point, we consider two distinct scenarios.

1) *Finite variance case:* We first assume that the second moment $m_2 = \int_{\mathbb{R}} a^2 V(da)$ of the Lévy measure V in (3) is finite. This allows us to rewrite the classical Lévy-Khinchine representation as

$$f(u) = jc_1 u - \frac{b_2 u^2}{2} + \int_{\mathbb{R} \setminus \{0\}} [e^{jau} - 1 - jua] V(da)$$

²A random variable x with PDF $p(x)$ is said to be infinitely divisible (id) if for any $n \in \mathbb{N}^+$ there exist i.i.d. random variables x_1, \dots, x_n with PDF say $p_n(x)$ such that x has the same distribution as $x_1 + \dots + x_n$.

where the Poisson part of the functional is now fully compensated. Indeed, we are guaranteed that the above integral is convergent because $|e^{jau} - 1 - jua| = O(a^2)$ as $a \rightarrow 0$ and $|e^{jau} - 1 - jua| = O(|a|)$ as $a \rightarrow \pm\infty$. An interesting non-Poisson example of infinitely-divisible probability laws that falls into this category (with non-finite V) is the Laplace density with Lévy triplet $(0, 0, V(da) = \frac{e^{-|a|}}{|a|} da)$ and $p(x) = \frac{1}{2}e^{-|x|}$. This model is particularly relevant for sparse signal processing because it provides a tight connection between Lévy processes and total variation regularization [24, Section VI].

Now, if the Lévy measure is finite $\int_{\mathbb{R}} V(da) = \lambda < \infty$, the admissibility condition yields $\int_{\mathbb{R}} a V(da) < \infty$, which allows us to pull the bias correction out of the integral. The representation then simplifies to

$$f(u) = jb_1u - \frac{b_2u^2}{2} + \lambda \int_{\mathbb{R}} [e^{jau} - 1] p_a(a) da,$$

with $V(da) = \lambda p_a(a) da$ and $\int_{\mathbb{R}} p_a(a) da = 1$. This implies that we can decompose x into the sum of two independent Gaussian and Compound-Poisson random variables. The variances of the Gaussian and Poisson components are $\sigma^2 = b_2$ and $\lambda E\{a^2\}$, respectively. The Poisson component is sparse because its probability density function exhibits a mass distribution $e^{-\lambda}\delta(x)$ at the origin, meaning that the chances for a continuous amplitude distribution of getting zero are overwhelmingly higher than any other value, especially for smaller values of $\lambda > 0$. It is therefore justifiable to use $0 \leq e^{-\lambda} < 1$ as our Poisson sparsity index.

2) *Infinite variance case:* We now turn our attention to the case where the second moment of the Lévy measure is unbounded, which we like to label as the “super-sparse” one. To substantiate this claim, we invoke the Ramachandran-Wolfe theorem which states that the p th moment $\mathcal{E}\{|x|^p\}$ with $p \in \mathbb{R}^+$ of an infinitely divisible distribution is finite iff. $\int_{|a|>1} |a|^p V(da) < \infty$ [35], [36]. For $p \geq 2$, the latter is equivalent to $\int_{\mathbb{R}} |a|^p V(da) < \infty$ because of the Lévy admissibility condition. It follows that the cases that are not covered by the previous scenario (including the Gaussian + Poisson model) necessarily give rise to distributions whose moments of order p are unbounded for $p \geq 2$. The prototypical representatives of such heavy tail distributions are the alpha-stable ones or, by extension, the broad family of infinite divisible probability laws that are in their domain of attraction. It has been shown that these distributions precisely fulfill the requirement for ℓ_p compressibility [37], which is a stronger form of sparsity than the presence of a mass probability density at $x = 0$.

III. INNOVATION APPROACH TO CONTINUOUS-TIME STOCHASTIC PROCESSES

Specifying a stochastic process through an innovation model (or an equivalent stochastic differential equation) is attractive conceptually, but it presupposes that we can provide an inverse operator (in the

form of an integral transform) that transforms the white noise back into the initial stochastic process. This is the reason why we will spend the greater part of our effort investigating suitable inverse operators.

A. Stochastic differential equations

Our aim is to define the generalized process with whitening operator L and noise parameters (b_1, b_2, V) as the solution of the stochastic linear differential equation

$$Ls = w, \quad (8)$$

where w is a white noise process, as described in subsection II-C. This definition is obviously only usable if we can construct an inverse operator $T = L^{-1}$ that solves this equation. For the cases where the inverse is not unique, we will need to select one preferential operator, which is equivalent to imposing specific boundary conditions. We are then able to formally express the stochastic process as a transformed version of a white noise

$$s = L^{-1}w. \quad (9)$$

The requirement for such a solution to be consistent with (8) is that the operator satisfies the right-inverse property $LL^{-1} = I$ over the underlying class of tempered distributions. By using the adjoint relation $\langle s, \varphi \rangle = \langle L^{-1}w, \varphi \rangle = \langle w, L^{-1*}\varphi \rangle$, we can then transfer the action of the operator onto the test function inside the characteristic form and obtain a complete statistical characterization of the so-defined generalized stochastic process

$$\mathcal{Z}_s(\varphi) = \mathcal{Z}_{L^{-1}w}(\varphi) = \mathcal{Z}_w(L^{-1*}\varphi), \quad (10)$$

where \mathcal{Z}_w is given by (2) (or one of the specific forms in the list at the end of section II-C) and where we are implicitly assuming that the adjoint L^{-1*} is mathematically well-defined over \mathcal{S} .

In order to fulfill the above mathematical requirements, it is usually easier to proceed backwards: one specifies an operator T that satisfies the left-inverse property: $\forall \varphi \in \mathcal{S}, TL^*\varphi = \varphi$, and that is continuous (i.e., bounded in a proper metric) over the chosen class of test functions. One then characterizes the adjoint of T , which, for a given $\phi \in \mathcal{S}$, is such that

$$\forall \varphi \in \mathcal{S}, \quad \langle T\varphi, \phi \rangle = \langle \varphi, T^*\phi \rangle.$$

Finally, one applies a standard limit argument to extend the action of $T^* = L^{-1}$ over the enlarged class of tempered distribution $\phi \in \mathcal{S}'$ based on the above adjoint relation, which yields the proper distributional definition of the right inverse of L in (9).

B. Inverse operators

Before presenting our general method of solution, we need to identify a suitable set of elementary inverse operators that satisfy the required boundedness conditions.

Our approach relies on the factorization of a differential operator into simple first-order components of the form $(D - \alpha_n \text{Id})$ with $\alpha_n \in \mathbb{C}$, which can then be treated separately. Three possible cases need to be considered.

1) *Causal-stable*: $\text{Re}(\alpha_n) < 0$. This is the classical textbook hypothesis which leads to a causal-stable convolution system. It is well known from linear system theory that the causal Green function of $(D - \alpha_n \text{Id})$ is

$$\rho_{\alpha_n}(t) = u(t)e^{\alpha_n t},$$

where $u(t) = 1_{[0, +\infty)}(t)$ is the unit-step (or Heaviside) function. Clearly, $\rho_{\alpha_n}(t)$ is absolutely integrable (and rapidly-decaying) iff. $\text{Re}(\alpha_n) < 0$. It follows that $(D - \alpha_n \text{Id})^{-1}f = \rho_{\alpha_n} * f$ with $\rho_{\alpha_n} \in \mathcal{R} \subset L_1$. In particular, this implies that $T = (D - \alpha_n \text{Id})^{-1}$ specifies a continuous LSI operator on \mathcal{S} . The same holds for $T^* = (D - \alpha_n \text{Id})^{-1*}$, which is defined as $T^*f = \rho_{\alpha_n}^\vee * f$.

2) *Anti-causal stable*: $\text{Re}(\alpha_n) > 0$. This case is usually excluded because the standard Green function $\rho_{\alpha_n}(t) = u(t)e^{\alpha_n t}$ grows exponentially, meaning that the system does not have a stable causal solution. Yet, it is possible to consider an alternative anti-causal Green function $\rho'_{\alpha_n}(t) = -\rho_{-\alpha_n}^\vee(t) = \rho_{\alpha_n}(t) - e^{\alpha_n t}$, which is unique in the sense that it is the only Green function³ of $(D - \alpha_n \text{Id})$ that is Lebesgue-integrable and, by the same token, the proper inverse Fourier transform of $\frac{1}{j\omega - \alpha_n}$ for $\text{Re}(\alpha_n) > 0$. In this way, we are able to specify an anti-causal inverse filter $(D - \alpha_n \text{Id})^{-1}f = \rho'_{\alpha_n} * f$ with $\rho'_{\alpha_n} \in \mathcal{R}$ that is L_p -stable and \mathcal{S} -continuous. In the sequel, we will drop the ' superscript with the convention that $\rho_{\alpha_n}(t)$ systematically refers to the unique Green function of $(D - \alpha_n \text{Id})$ that is in \mathcal{R} (rapidly-decaying); it is either causal or anti-causal depending on the polarity of $\text{Re}(\alpha_n)$.

3) *Marginally stable*: $\text{Re}(\alpha_n) = 0$ or, equivalently, $\alpha_n = j\omega_0$ with $\omega_0 \in \mathbb{R}$. This third case, which is incompatible with the conventional formulation of stationary processes, is most interesting theoretically because it opens the door to important extensions such as Lévy processes and fractals (fractional Brownian motion). Here, we will show that marginally-stable systems can be handled within our generalized framework as well, thanks to the introduction of appropriate inverse operators.

³: ρ is a Green functions of $(D - \alpha_n \text{Id})$ iff. $(D - \alpha_n \text{Id})^{-1}\rho = \delta$; the complete set of solutions is given $\rho(t) = \rho_{\alpha_n}(t) + Ce^{\alpha_n t}$ which is the sum of the causal Green function $\rho_{\alpha_n}(t)$ plus an arbitrary exponential component that is in the null space of the operator.

The first natural candidate for $(D - j\omega_0 \text{Id})^{-1}$ is the inverse filter whose frequency response is

$$\hat{\rho}_{j\omega_0}(\omega) = \frac{1}{j(\omega - \omega_0)} + \pi\delta(\omega - \omega_0).$$

It is a convolution operator whose time-domain definition is

$$\begin{aligned} \mathbf{I}_{\omega_0}\varphi(t) &= (\rho_{j\omega_0} * \varphi)(t) \\ &= e^{j\omega_0 t} \int_{-\infty}^t e^{-j\omega_0 \tau} \varphi(\tau) d\tau \end{aligned} \quad (11)$$

where $\rho_{j\omega_0}(t) = u(t)e^{j\omega_0 t}$; its adjoint is given by

$$\begin{aligned} \mathbf{I}_{\omega_0}^* \varphi(t) &= (\rho_{j\omega_0}^\vee * \varphi)(t) \\ &= e^{-j\omega_0 t} \int_t^{+\infty} e^{j\omega_0 \tau} \varphi(\tau) d\tau. \end{aligned} \quad (12)$$

While $\mathbf{I}_{\omega_0}\varphi(t)$ and $\mathbf{I}_{\omega_0}^* \varphi(t)$ are both well-defined pointwise when $\varphi \in L_1$, the problem is that these inverse filters are not BIBO stable since their impulse responses, $\rho_{j\omega_0}(t)$ and $\rho_{j\omega_0}^\vee(t)$, are not in L_1 . In particular, one can easily see that $\mathbf{I}_{\omega_0}\varphi$ (resp., $\mathbf{I}_{\omega_0}^* \varphi$) with $\varphi \in \mathcal{S}$ is generally not in L_p with $1 \leq p < +\infty$, unless $\hat{\varphi}(\omega_0) = 0$ (resp., $\hat{\varphi}(-\omega_0) = 0$). The conclusion is that $\mathbf{I}_{\omega_0}^*$ fails to be a bounded operator over the class of test functions \mathcal{S} .

This leads us to introduce some “corrected” version of the adjoint inverse operator $\mathbf{I}_{\omega_0}^*$

$$\begin{aligned} \mathbf{I}_{\omega_0, t_0}^* \varphi(t) &= \mathbf{I}_{\omega_0}^* \{ \varphi - \hat{\varphi}(-\omega_0) e^{-j\omega_0 t_0} \delta(\cdot - t_0) \} (t) \\ &= \mathbf{I}_{\omega_0}^* \varphi(t) - \hat{\varphi}(-\omega_0) e^{-j\omega_0 t_0} \rho_{j\omega_0}^\vee(t - t_0), \end{aligned} \quad (13)$$

where $t_0 \in \mathbb{R}$ is a fixed location parameter and where $\hat{\varphi}(-\omega_0) = \int_{\mathbb{R}} e^{j\omega_0 t} \varphi(t) dt$ is the complex sinusoidal moment associated with the frequency ω_0 . The idea is to correct for the lack of decay of $\mathbf{I}_{\omega_0}^* \varphi(t)$ as $t \rightarrow -\infty$ by subtracting a properly weighted version of the impulse response of the operator. An equivalent Fourier-based formulation is provided by the formula at the bottom of Table I; the main difference with the corresponding expression for $\mathbf{I}_{\omega_0}\varphi$ is the presence of a regularization term in the numerator that prevents the integrand from diverging at $\omega = \omega_0$. The next step is to identify the adjoint of $\mathbf{I}_{\omega_0, t_0}^*$, which is achieved via the following inner-product manipulation

$$\begin{aligned} \langle \varphi, \mathbf{I}_{\omega_0, t_0}^* \phi \rangle &= \langle \varphi, \mathbf{I}_{\omega_0}^* \phi \rangle - \underbrace{\hat{\varphi}(-\omega_0) e^{-j\omega_0 t_0} \langle \varphi, \rho_{j\omega_0}^\vee(\cdot - t_0) \rangle}_{\text{(by linearity)}} \\ &= \langle \mathbf{I}_{\omega_0} \varphi, \phi \rangle - \langle e^{j\omega_0 t}, \phi \rangle e^{-j\omega_0 t_0} \mathbf{I}_{\omega_0} \varphi(t_0) && \text{(using (11))} \\ &= \langle \mathbf{I}_{\omega_0} \varphi, \phi \rangle - \langle e^{j\omega_0(t-t_0)} \mathbf{I}_{\omega_0} \varphi(t_0), \phi \rangle. \end{aligned}$$

Since the above is equal to $\langle I_{\omega_0, t_0} \varphi, \phi \rangle$ by definition, we obtain that

$$I_{\omega_0, t_0} \varphi(t) = I_{\omega_0} \varphi(t) - e^{j\omega_0(t-t_0)} I_{\omega_0} \varphi(t_0). \quad (14)$$

Interestingly, this operator imposes the boundary condition $I_{\omega_0, t_0} \varphi(t_0) = 0$ via the subtraction of a sinusoidal component that is in the null space of the operator $(D - j\omega_0 \text{Id})$, which gives a direct interpretation of the location parameter t_0 . Observe that expressions (13) and (14) define linear operators, albeit not shift-invariant ones, in contrast with the classical inverse operators I_{ω_0} and $I_{\omega_0}^*$.

For analysis purposes, it is convenient to relate the proposed inverse operators to the anti-derivatives corresponding to the case $\omega_0 = 0$. To that end, we introduce the modulation operator

$$M_{\omega_0} \varphi(t) = e^{j\omega_0 t} \varphi(t)$$

which is a unitary map on L_2 with the property that $M_{\omega_0}^{-1} = M_{-\omega_0}$.

Proposition 1: The inverse operators defined by (11), (12), (14), and (13) satisfy the modulation relations

$$\begin{aligned} I_{\omega_0} \varphi(t) &= M_{\omega_0} I_0 M_{\omega_0}^{-1} \varphi(t), \\ I_{\omega_0}^* \varphi(t) &= M_{\omega_0}^{-1} I_0^* M_{\omega_0} \varphi(t), \\ I_{\omega_0, t_0} \varphi(t) &= M_{\omega_0} I_{0, t_0} M_{\omega_0}^{-1} \varphi(t), \\ I_{\omega_0, t_0}^* \varphi(t) &= M_{\omega_0}^{-1} I_{0, t_0}^* M_{\omega_0} \varphi(t). \end{aligned}$$

Proof: These follow from the modulation property of the Fourier transform (i.e., $\mathcal{F}\{M_{\omega_0} \varphi\}(\omega) = \mathcal{F}\{\varphi\}(\omega - \omega_0)$) and the observations that $I_{\omega_0} \delta(t) = \rho_{j\omega_0}(t) = M_{\omega_0} \rho_0(t)$ and $I_{\omega_0}^* \delta(t) = \rho_{j\omega_0}^\vee(t) = M_{-\omega_0} \rho_0^\vee(t)$ with $\rho_0(t) = u(t)$ (unit step). ■

The important functional property of I_{ω_0, t_0}^* is that it essentially preserves decay and integrability, while I_{ω_0, t_0} fully retains signal differentiability. Unfortunately, it is not possible to have the two simultaneously unless $I_{\omega_0} \varphi(t_0)$ and $\hat{\varphi}(-\omega_0)$ are both zero.

Proposition 2: If $f \in L_\infty(\mathbb{R}, w_\alpha)$ with $\alpha > 1$, then there exists a constant C_{t_0} such that

$$|I_{\omega_0, t_0}^* f(t)| \leq C_{t_0} \frac{\|f\|_{\infty, \alpha}}{1 + |t|^{\alpha-1}},$$

which implies that $I_{\omega_0, t_0}^* f \in L_{\infty, \alpha-1}$.

Proof: Since modulation does not affect the decay properties of a function, we can invoke Proposition 1 and concentrate on the investigation of the anti-derivative operator I_{0, t_0}^* . Without loss of generality, we

can also pick $t_0 = 0$ and transfer the bound to any other finite value of t_0 by adjusting the value of the constant C_{t_0} . Specifically, for $t < 0$, we write this inverse operator as

$$\begin{aligned} \mathbf{I}_{0,0}^* f(t) &= \mathbf{I}_0^* f(t) - \hat{f}(0) \\ &= \int_t^{+\infty} f(\tau) \, d\tau - \int_{-\infty}^{\infty} f(\tau) \, d\tau \\ &= - \int_{-\infty}^t f(\tau) \, d\tau. \end{aligned}$$

This implies that

$$|\mathbf{I}_{0,0}^* f(t)| = \left| \int_{-\infty}^t f(\tau) \, d\tau \right| \leq \|f\|_{\infty, \alpha} \int_{-\infty}^t \frac{1}{1 + |\tau|^\alpha} \, d\tau \leq \left(\frac{2\alpha}{\alpha - 1} \right) \frac{\|f\|_{\infty, \alpha}}{1 + |t|^{\alpha-1}}.$$

For $t > 0$, $\mathbf{I}_{0,0}^* f(t) = \int_t^{\infty} f(\tau) \, d\tau$ so that the above upper bounds remain valid. \blacksquare

The interpretation of the above result is that the inverse operator $\mathbf{I}_{\omega_0, t_0}^*$ reduces inverse polynomial decay by one order. Proposition 2 actually implies that the operator will preserve the rapid decay of the Schwartz functions which are included in $L_{\infty, \alpha}$ for any $\alpha \in \mathbb{R}^+$. It also guarantees that $\mathbf{I}_{\omega_0, t_0}^* \varphi$ belongs to L_p for any Schwartz function φ . However, $\mathbf{I}_{\omega_0, t_0}^*$ will spoil the global smoothness properties of φ because it introduces a discontinuity at t_0 , unless $\hat{\varphi}(-\omega_0)$ is zero in which case the output remains in the Schwartz class. This allows us to state the following theorem which summarizes the higher-level part of those results for further reference.

Theorem 1: The operator $\mathbf{I}_{\omega_0, t_0}^*$ defined by (14) is a continuous linear map from \mathcal{S} into \mathcal{R} (the space of bounded functions with rapid decay). Its adjoint $\mathbf{I}_{\omega_0, t_0}$ is given by (13) and has the property that $\mathbf{I}_{\omega_0, t_0} \varphi(t_0) = 0$. Together, these operators satisfy the complementary left- and right-inverse relations

$$\begin{aligned} \mathbf{I}_{\omega_0, t_0}^* (\mathbf{D} - j\omega_0 \text{Id})^* \varphi &= \varphi \\ (\mathbf{D} - j\omega_0 \text{Id}) \mathbf{I}_{\omega_0, t_0} \varphi &= \varphi \end{aligned}$$

for all $\varphi \in \mathcal{S}$.

Having a tight control on the action of $\mathbf{I}_{\omega_0, t_0}^*$ over \mathcal{S} allows us to extend the right-inverse operator $\mathbf{I}_{\omega_0, t_0}$ to an appropriate subset of tempered distributions $\phi \in \mathcal{S}'$ according to the rule $\langle \mathbf{I}_{\omega_0, t_0} \phi, \varphi \rangle = \langle \phi, \mathbf{I}_{\omega_0, t_0}^* \varphi \rangle$. Our complete set of inverse operators is summarized in Table I together with their equivalent Fourier-based definitions which are also interpretable in the generalized sense of distributions.

C. Solution of generic stochastic differential equation

We now have all the elements to solve the generic stochastic linear differential equation

$$\sum_{n=1}^N a_n \mathbf{D}^n s = \sum_{m=1}^M b_m \mathbf{D}^m w \quad (15)$$

TABLE I
FIRST-ORDER DIFFERENTIAL OPERATORS AND THEIR INVERSES

L	$L^{-1}f(t)$	Properties of inverse operator
Standard case: $\alpha_n \in \mathbb{C}, \text{Re}(\alpha_n) \neq 0$		
$(D - \alpha_n \text{Id})$	$(D - \alpha_n \text{Id})^{-1}f(t) = \int_{\mathbb{R}} \hat{f}(\omega) \left(\frac{1}{j\omega - \alpha_n} \right) e^{j\omega t} \frac{d\omega}{2\pi}$	L_p -stable, LSI, \mathcal{S} -continuous
$(D - \alpha_n \text{Id})^*$	$(D^* - \alpha_n \text{Id})^{-1}f(t) = \int_{\mathbb{R}} \hat{f}(\omega) \left(\frac{1}{-j\omega - \alpha_n} \right) e^{j\omega t} \frac{d\omega}{2\pi}$	L_p -stable, LSI, \mathcal{S} -continuous
Critical case: $\alpha_n = j\omega_0, \omega_0 \in \mathbb{R}$		
$(D - j\omega_0 \text{Id})$	$I_{\omega_0}f(t) = \int_{\mathbb{R}} \hat{f}(\omega) \left(\frac{1}{j(\omega - \omega_0)} + \pi\delta(\omega - \omega_0) \right) e^{j\omega t} \frac{d\omega}{2\pi}$	Causal, LSI
	$I_{\omega_0, t_0}f(t) = \int_{\mathbb{R}} \hat{f}(\omega) \left(\frac{e^{j\omega t} - e^{j\omega_0(t-t_0)}e^{j\omega t_0}}{j(\omega - \omega_0)} \right) \frac{d\omega}{2\pi}$	Output vanishes at $t = t_0$
$(D - j\omega_0 \text{Id})^*$	$I_{\omega_0}^*f(t) = \int_{\mathbb{R}} \left(\frac{\hat{f}(\omega)}{-j(\omega + \omega_0)} + \hat{f}(-\omega_0)\pi\delta(\omega + \omega_0) \right) e^{j\omega t} \frac{d\omega}{2\pi}$	Anti-causal, LSI
	$I_{\omega_0, t_0}^*f(t) = \int_{\mathbb{R}} \left(\frac{\hat{f}(\omega) - \hat{f}(-\omega_0)e^{-j(\omega + \omega_0)t_0}}{-j(\omega + \omega_0)} \right) e^{j\omega t} \frac{d\omega}{2\pi}$	L_p -stable and decay preserving

where the a_n and b_m are arbitrary complex coefficients with the normalization constraint $a_N = 1$. While this reminds us of the textbook formula of an ordinary N th-order differential system, the non-standard aspect here is that the driving term is a white noise process w , which is generally not defined pointwise, and that we are not imposing any stability constraint. Eq. (15) thus covers the general case (8) where L is a shift-invariant operator with the rational transfer function

$$\hat{L}(\omega) = \frac{(j\omega)^N + a_{N-1}(j\omega)^{N-1} + \dots + a_1(j\omega) + a_0}{b_M(j\omega)^M + \dots + b_1(j\omega) + b_0} = \frac{P_N(j\omega)}{Q_M(j\omega)}. \quad (16)$$

The poles of the system, which are the roots of the characteristic polynomial $P_N(\zeta) = \zeta^N + a_{N-1}\zeta^{N-1} + \dots + a_0$ with Laplace variable $\zeta \in \mathbb{C}$, are denoted by $\{\alpha_n\}_{n=1}^N$. While we are not imposing any restriction on their locus in the complex plane, we are adopting a special ordering where the purely imaginary roots (if present) are coming last. This allows us to factorize the numerator of (16) as

$$P_N(j\omega) = \prod_{n=1}^N (j\omega - \alpha_n) = \left(\prod_{n=1}^{N-n_0} (j\omega - \alpha_n) \right) \left(\prod_{m=1}^{n_0} (j\omega - j\omega_m) \right) \quad (17)$$

with $\alpha_{N-n_0+m} = j\omega_m$ where n_0 is the number of purely-imaginary poles. The operator counterpart of this last equation is the decomposition

$$P_N(D) = \underbrace{(D - \alpha_1 \text{Id}) \cdots (D - \alpha_{N-n_0} \text{Id})}_{\text{regular part}} \underbrace{(D - j\omega_1 \text{Id}) \cdots (D - j\omega_{n_0} \text{Id})}_{\text{critical part}}$$

which involves a cascade of elementary first-order components. By applying the proper sequence of right-inverse operators from Table 1, we can then formally solve the system as in (9). The resulting inverse operator is

$$L^{-1} = \underbrace{I_{\omega_{n_0}, t_{n_0}} \cdots I_{\omega_1, t_1}}_{\text{shift-variant}} T_{\text{LSI}} \quad (18)$$

with

$$T_{\text{LSI}} = (D - \alpha_{N-n_0} \text{Id})^{-1} \cdots (D - \alpha_1 \text{Id})^{-1} Q_M(D),$$

which imposes the n_0 boundary conditions

$$\left\{ \begin{array}{ll} s(t)|_{t=t_{n_0}} & = 0 \\ (D - j\omega_{n_0} \text{Id})s(t)|_{t=t_{n_0-1}} & = 0 \\ \vdots & \\ (D - j\omega_2 \text{Id}) \cdots (D - j\omega_{n_0} \text{Id})s(t)|_{t=t_1} & = 0. \end{array} \right. \quad (19)$$

The corresponding adjoint operator is given by

$$L^{-1*} = T_{\text{LSI}}^* \underbrace{I_{\omega_1, t_1}^* \cdots I_{\omega_{n_0}, t_{n_0}}^*}_{\text{shift-variant}}, \quad (20)$$

and is guaranteed to be a continuous linear mapping from \mathcal{S} into \mathcal{R} by Theorem 1, the key point being that each of the component operators preserves the rapid decay of the test function to which it is applied. The last step is to substitute the explicit form (20) of L^{-1*} into (10), which yields the characteristic form of the stochastic process s defined by (15) subject to the boundary conditions (19).

We close this section with a comment about commutativity: while the order of application of the operators $Q_M(D)$ and $(D - \alpha_n \text{Id})^{-1}$ in the LSI part of (18) is immaterial (thanks to the commutativity of convolution), it is not so for the inverse operators I_{ω_m, t_0} that appear in the “shift-variant” part of the decomposition. The latter do not commute and their order of application is tightly linked to the boundary conditions.

IV. PROPERTIES OF GENERALIZED STOCHASTIC PROCESSES

A. Green functions and exponential B-spline calculus

The foundation of the exponential spline calculus is that we can always factor an N th-order differential operator into a cascade of first-order operators $P_{\alpha_n} = (D - \alpha_n \text{Id})$ where the α_n (complex poles) are the roots of the characteristic polynomial; i.e.,

$$\begin{aligned} P_N(D) &= D^N + a_{N-1}D^{N-1} + \cdots + a_1D + a_0\text{Id} \\ &= P_{\alpha_N} \cdots P_{\alpha_1} = P_{(\alpha_1, \dots, \alpha_N)} \end{aligned}$$

where the right-hand side concatenated operator notation is self-explanatory. This allows us to express the Green function of P_{α} with pole vector $\alpha = (\alpha_1, \dots, \alpha_N)$ as the convolution of the Green functions of its elementary constituents

$$\rho_{\alpha}(t) = (\rho_{\alpha_1} * \rho_{\alpha_2} \cdots * \rho_{\alpha_N})(t) \quad (21)$$

with

$$\rho_{\alpha}(t) = \begin{cases} u(t)e^{\alpha t} & \text{if } \text{Re}(\alpha) \leq 0 \\ -u(-t)e^{\alpha t} & \text{else.} \end{cases} \quad (22)$$

The so-defined Green function $\rho_{\alpha}(t)$ is necessarily of slow growth; it specifies the impulse response of the LSI inverse operator P_{α}^{-1} , which is well-defined over \mathcal{S} ,

$$P_{\alpha}^{-1}\varphi(t) = (\rho_{\alpha} * \varphi)(t),$$

but not necessarily bounded, as we have seen in the case of purely imaginary poles.

Next, we observe that by applying the finite difference operator

$$\Delta_{\alpha}f(t) = f(t) - e^{\alpha}f(t-1)$$

to the function $\rho_{\alpha}(t)$, we are able to construct a compactly-supported function: the first-order exponential B-spline with parameter α

$$\beta_{\alpha}(t) = \Delta_{\alpha}\rho_{\alpha}(t) = \begin{cases} 1_{[0,1)}(t)e^{\alpha t} & \text{if } \text{Re}(\alpha) \leq 0 \\ 1_{[0,1)}(t)e^{\alpha(t-1)} & \text{else.} \end{cases}$$

The generalization of this scheme yields the N th-order B-spline with parameter vector $\alpha = (\alpha_1, \dots, \alpha_N)$

$$\beta_{\alpha}(t) = \Delta_{\alpha}\rho_{\alpha}(t) = (\beta_{\alpha_1} * \beta_{\alpha_2} \cdots * \beta_{\alpha_N})(t). \quad (23)$$

These functions have the following properties:

- They are smooth and well-localized: compactly supported in $[0, N]$, bounded, and Hölder continuous of order $N - 1$.
- They are piecewise-exponential with joining points at the integer and a maximal degree of smoothness (spline property). For $\alpha = (0, \dots, 0)$, one recovers Schoenberg's classical polynomial B-splines of degree $N - 1$ [38], [39].
- They are the shortest elementary constituents of splines: the functions $\{\beta_\alpha(t - n)\}_{n \in \mathbb{Z}}$ forms a Riesz basis of the corresponding family of exponential splines with knots at the integers.

The crucial formula for our purpose is the equivalent operator interpretation of the B-spline formula (23):

$$\Delta_\alpha P_\alpha^{-1} \varphi = \Delta_\alpha \rho_\alpha * \varphi = \beta_\alpha * \varphi, \quad (24)$$

which we will now put to good use in order to stationarize and localize the effect of the inverse operators that were encountered in the previous sections.

Theorem 2: Let $\{I_{\omega_m, t_m}^*\}_{m=1}^{n_0}$ with $\omega_m \in \mathbb{R}$ be a series of operators of the type defined by (13) and let $\{\Delta_{j\omega_m}^*\}_{m=1}^{n_0}$ be some corresponding adjoint localization operators with $\Delta_{j\omega_m}^* \varphi(t) = \varphi(t) - e^{j\omega_m} \varphi(t+1)$. Then,

$$\begin{aligned} I_{\omega_1, t_1}^* \cdots I_{\omega_{n_0}, t_{n_0}}^* \Delta_{j\omega_{n_0}}^* \cdots \Delta_{j\omega_1}^* \varphi &= \beta_{(j\omega_1, \dots, j\omega_{n_0})}^\vee * \varphi \\ \Delta_{j\omega_1}^* \cdots \Delta_{j\omega_{n_0}}^* I_{\omega_{n_0}, t_{n_0}} \cdots I_{\omega_1, t_1} \varphi &= \beta_{(j\omega_1, \dots, j\omega_{n_0})} * \varphi \end{aligned}$$

for all $\varphi \in \mathcal{S}$, where $\beta_{(j\omega_1, \dots, j\omega_{n_0})}$ an exponential B-spline kernel as defined by (23). Since the latter is bounded and compactly-supported, the resulting convolution operators are BIBO-stable and \mathcal{S} -continuous.

Proof: First, we observe that $\widehat{\Delta_{j\omega_m}^* f}(\omega) = (1 - e^{j\omega_m} e^{j\omega}) \hat{f}(\omega)$. Using the bottom formula of Table I, we then evaluate the Fourier transform of $g(t) = I_{\omega_m, t_m}^* \Delta_{j\omega_m}^* f(t)$ as

$$\begin{aligned} \hat{g}(\omega) &= \frac{(1 - e^{j\omega_m} e^{j\omega}) \hat{f}(\omega) - \overbrace{(1 - e^{j\omega_m} e^{-j\omega_m})}^{=0} \hat{f}(-\omega_m) e^{-j(\omega + \omega_m)t_m}}{-j(\omega + \omega_m)} \\ &= \hat{f}(\omega) \left(\frac{1 - e^{j\omega_m + j\omega}}{-j\omega - j\omega_m} \right), \end{aligned}$$

where we identify the right-hand side factor as $\hat{\beta}_{j\omega_m}(-\omega)$ where $\hat{\beta}_\alpha(\omega) = \frac{1 - e^{\alpha - j\omega}}{j\omega - \alpha}$ is the Fourier transform of the first-order exponential B-spline with parameter α . This proves that $I_{\omega_m, t_m}^* \Delta_{j\omega_m}^* f = \beta_{j\omega_m}^\vee * f$ for any $\omega_m, t_m \in \mathbb{R}$. Using the property that the order of application of stable convolution operators such as $\Delta_{j\omega_m}^*$ can be changed (commutativity), we start with $I_{\omega_{n_0}, t_{n_0}}^* \Delta_{j\omega_{n_0}}^* f$ and progressively work our way outwards to show that $I_{\omega_1, t_1}^* \cdots I_{\omega_{n_0}, t_{n_0}}^* \Delta_{j\omega_{n_0}}^* \cdots \Delta_{j\omega_1}^* \varphi = \beta_{j\omega_1}^\vee * \cdots * \beta_{j\omega_{n_0}}^\vee * \varphi$, which, thanks to (23), yields the desired result. The second formula is established in the same way. ■

The interpretation of the second relation is that the difference operators $\Delta_{j\omega_n}$ annihilate the sinusoidal components that are in the null space of $(D - j\omega_n)$ so that the effect of I_{ω_m, t_m} becomes indistinguishable from that of the non-regularized shift-invariant inverse I_{ω_m} . By combining this result with (24), we obtain a stable LSI substitute for the original inverse operator with the added benefit of a much better localization.

Corollary 1: Let L^{-1*} be the N th-order (not necessarily shift-invariant) inverse operator specified by (20). Then,

$$\begin{aligned} L^{-1*} \Delta_{\alpha}^* \varphi &= \beta_L^{\vee} * \varphi \\ \Delta_{\alpha} L^{-1} \varphi &= \beta_L * \varphi, \end{aligned}$$

where β_L is the generalized B-spline kernel

$$\beta_L(t) = Q_M(D) \beta_{\alpha}(t) = \sum_{m=1}^M b_m D^m \beta_{\alpha}(t). \quad (25)$$

The latter is a linear combination of derivatives of the N th-order exponential B-spline $\beta_{\alpha}(t)$ with parameter vector $\alpha = (\alpha_1, \dots, \alpha_N)$, and is therefore compactly-supported over the time-interval $[0, N]$.

The intuition behind this result is that we are localizing the system's response by canceling the poles of its frequency response; i.e., a pole at $j\omega = \alpha_n$ is neutralized by a corresponding zero of $1 - e^{\alpha_n - j\omega}$ (the frequency response of Δ_{α_n}).

B. Characterization of generalized stochastic processes

We now proceed with the explicit characterization of a broad class of stochastic processes governed by the innovation model (8). Besides the fundamental issue of the solvability of such an operator equation which has already been addressed in Section III-C, one needs to make sure that the solutions are bona fide generalized stochastic processes. The answer, of course, is dependent upon whether or not we are able to exhibit an (adjoint) inverse operator $T = L^{-1*}$ that is sufficiently well-behaved for the resulting characteristic form $\mathcal{Z}_s(\varphi) = \mathcal{Z}_w(T\varphi)$ to be admissible. To that end, we can rely on the following theorem whose proof is given in Appendix II.

Theorem 3 (Admissibility): Let $\mathcal{Z}_s(\varphi) = e^{\int_{\mathbb{R}} f(T\varphi) dt}$ where the scalar function $f(u)$ is of the generic Lévy-Khintchine form (3) and where T is an operator acting on $\varphi \in \mathcal{S}$. Then, $\mathcal{Z}_s(\varphi)$ is a continuous, positive-definite functional on \mathcal{S} such that $\mathcal{Z}_s(0) = 1$, provided that any one of the conditions below is met:

- 1) T is a continuous linear map from \mathcal{S} into itself,

- 2) T is a continuous linear map from \mathcal{S} into L_p and $f(u)$ is such that $|f(u)| + |u| \cdot |f'(u)| \leq C|u|^p$ for all $u \in \mathbb{R}$, where $1 \leq p < \infty$ and C is a positive constant.

The simplest scenario is when L^{-1} is LSI and can be decomposed into a cascade of BIBO-stable and ordinary differential operators. If the BIBO-stable part is rapidly-decreasing, then L^{-1} is guaranteed to be \mathcal{S} -continuous. In particular, this covers the case of an N th-order differential system without any pole on the imaginary axis, as justified by our analysis in Section III-C.

Property 1 (Generalized stationary processes): Let L^{-1} (the right-inverse of some operator L) be a \mathcal{S} -continuous convolution operator characterized by its impulse response $\rho_L = L^{-1}\delta$. Then, the generalized stochastic processes that are defined by $\mathcal{Z}_s(\varphi) = \exp\left(\int_{\mathbb{R}} f(\rho_L^\vee * \varphi) dt\right)$ where $f(u)$ is of the generic form (3) are *stationary* and well-defined solutions of the operator equation (8) driven by some corresponding white noise process w .

Proof: The fact that these generalized processes are well-defined is a direct consequence of the Minlos-Bochner Theorem since L^{-1*} (the convolution with ρ_L^\vee) satisfies the strictest version of admissibility in Theorem 3. The stationarity property is equivalent to $\mathcal{Z}_s(\varphi) = \mathcal{Z}(\varphi(\cdot - t_0))$ for all $t_0 \in \mathbb{R}$; it is established by simple change of variable in the inner integral using the basic shift-invariance property of convolution; i.e., $(\rho_L^\vee * \varphi(\cdot - t_0))(t) = (\rho_L^\vee * \varphi)(t - t_0)$. ■

The above characterization is not only remarkably concise, but also quite general. It extends the traditional theory of stationary Gaussian processes, which corresponds to the choice $f(u) = -\frac{\sigma_0^2}{2}u^2$. The Gaussian case results in the simplified form $\int_{\mathbb{R}} f(L^{-1*}\varphi) dt = -\frac{\sigma_0^2}{2}\|\rho_L^\vee * \varphi\|_{L_2}^2 = -\frac{1}{4\pi} \int_{\mathbb{R}} \Phi_s(\omega) |\hat{\varphi}(\omega)|^2 d\omega$ (using Parseval's identity) where $\Phi_s(\omega) = \frac{\sigma_0^2}{|\hat{L}(-\omega)|^2}$ is the spectral power density that is associated with the innovation model. The interest here is that we get access to a much broader family of non-Gaussian processes (e.g., generalized Poisson or alpha-stable) with matched spectral properties since they share the same whitening operator L .

The characteristic form condenses all the statistical information about the process. For instance, by setting $\varphi = \omega\delta(\cdot - t_0)$, we can explicitly determine $\mathcal{Z}_s(\varphi) = \mathcal{E}\{e^{j\langle s, \varphi \rangle}\} = \mathcal{E}\{e^{j\omega s(t_0)}\} = \mathcal{F}\{p(s(t_0))\}(-\omega)$, which yields the characteristic function of the first-order probability density of the sample values of the process: $p(s(t_0)) = p(s)$. In the present stationary scenario, we find that $p(s) = \mathcal{F}^{-1}\{\exp\left(\int_{\mathbb{R}} f(-\omega\rho_L(t)) dt\right)\}(s)$, which requires the evaluation of an integral followed by an inverse Fourier transform. While this type of calculation is only tractable analytically in special cases, it may be performed numerically with the help of the FFT. Higher-order density functions are accessible as well as at the cost of some multi-dimensional inverse Fourier transforms. The same applies for moments which can be obtained through a simpler differentiation process, as exemplified in Section IV-C.

The further reaching aspect of the present formulation is that it is also applicable to the characterization of non-stationary processes such as Brownian motion and Lévy processes, which are usually treated separately from the stationary ones, and that it naturally leads to the identification of a whole variety of higher-order extensions. The commonality is that these non-stationary processes can all be derived as solutions of an (unstable) N th-order differential equation with some poles on the imaginary axis. This corresponds to the setting in Section III-C with $n_0 > 0$.

Property 2 (Generalized N th-order Lévy processes): Let L^{-1} (the right-inverse of an N th-order differential operator L) be specified by (18) with at least one non-shift-invariant factor I_{ω_1, t_1} . Then, the generalized stochastic processes that are defined by $\mathcal{Z}_s(\varphi) = \exp\left(\int_{\mathbb{R}} f(L^{-1*}\varphi) dt\right)$, where $f(u)$ is of the generic form (3) subject to the constraint $|f(u)| + |u| \cdot |f'(u)| \leq C|u|^p$ for some $p \geq 1$, are well-defined solutions of the stochastic differential equation (15) driven by some corresponding Lévy white noise w . These processes satisfy the boundary conditions (19) and are *non-stationary*.

Note that the required condition on $f(u)$ is satisfied by the great majority of the members of the Lévy-Kintchine family. For instance in the Poisson case, we can show that $|u| \cdot |f'(u)| \leq \lambda|u|E\{|a|\}$ and $f(u) \leq \lambda|u|E\{|a|\}$ by using the fact $|e^{jx} - 1| \leq |x|$; this implies that the bound in Theorem 3 with $p = 1$ is always satisfied provided that the first (absolute) moment of the amplitude PDF $p_a(a)$ is finite. The only cases we are aware of that do not fulfill the condition are the alpha-stable noises with $0 < \alpha < 1$, which are notorious for their exotic behavior.

Proof: The result is a direct consequence of the analysis in Section III-C—in particular, Eqs. (18)–(20)—and Proposition 2. The latter implies that $L^{-1*}\varphi$ is bounded in all $L_{\infty, m}$ norms with $m > n_0$ where n_0 is the number of purely imaginary poles, as specified earlier. Since $\mathcal{S} \subset L_{\infty, m} \subset L_p$ for $m > 1$ and the Schwartz topology is the strongest in this chain, we can infer that L^{-1*} is a continuous operator from \mathcal{S} onto any of the L_p spaces with $p \geq 1$. The existence claim then follows from the combination of Theorem 3 and Minlos-Bochner. Since $L^{-1*}\varphi$ is not shift-invariant, there is no chance for these processes to be stationary, not to mention the fact that they need to fulfill the boundary conditions (19). ■

Conceptually, we like to view the generalized stochastic processes of Property 2 as “adjusted” versions of the stationary ones that include some additional sinusoidal (or polynomial) trends. While the generation mechanism of these trends is random, there is a deterministic aspect to it because it imposes the boundary conditions (19) at t_1, \dots, t_{n_0} . The class of such processes is actually quite rich and the formalism surprisingly powerful. We shall illustrate the use of Property 2 in Section V with the simplest possible operator $L = D$ which will gets us back to Brownian motion and the celebrated family of Lévy processes. We shall also show how the well-known properties of Lévy processes can be readily deduced from their

characteristic form.

Since the study of Lévy processes is usually perceived as presenting a greater level of difficulty than the classical theory of stationary processes, one could expect that the same applies to the generalized processes of Property 2 because: 1) they are solutions of unstable equations (poles on the imaginary axis), and 2) they are non-stationary, which complicates their statistical characterization. While this is indeed the case, there is an interesting, practical way to get around the main difficulty by taking advantage of the localizing properties of the finite difference operators Δ_α introduced in Section IV-A. These do actually provide the proper extension of the classical notion of increments for Brownian motion and Lévy processes.

Property 3 (Generalized increment processes): Let s be a generalized stochastic process whose characteristic form is $\mathcal{Z}_s(\varphi) = \mathcal{Z}_w(L^{-1*}\varphi)$ where \mathcal{Z}_w is a white noise functional and where L^{-1*} is given by (20) (differential system of order N with pole vector α and driving operator $Q_M(D) = \sum_{m=1}^M b_m D^m$). The corresponding generalized increment process

$$s_d(t) = \Delta_\alpha s(t)$$

is well-defined and stationary (irrespective of any stability consideration). Its characteristic form is given by $\mathcal{Z}_{s_d}(\varphi) = \mathcal{Z}_w(\beta_L^\vee * \varphi)$ where β_L is the generalized B-spline kernel defined by (25).

Proof: : Corollary 1 implies that $\Delta_\alpha L^{-1}w = \beta_L * w$. Since the convolution with the compactly-supported kernel β_L defines a continuous LSI operator on \mathcal{S} , we can invoke Property 1 with $\rho = \beta_L$, which yields the desired result. ■

Since the generalized B-spline β_L is Hölder-continuous of order $N - M - 1$, the above characterization allows us to infer that the two processes s_d and s are $(N - M - 2)$ times differentiable in the classical sense. In fact, the processes are well-defined pointwise as soon as $N > M$, which is the minimum requirement for continuity in the mean-square sense [40]. The other direct implication is that the samples of the generalized increment process, $s_d(t_1)$ and $s_d(t_2)$, are independent as soon as $|t_1 - t_2| > N$ (due to the finite support property of the exponential B-spline β_L). This means that working with the increment process $s_d(t)$ has the remarkable feature of completely suppressing long-range dependencies.

C. Moments and correlation

The covariance form of a generalized (complex-valued) process s is defined as:

$$\mathcal{B}_s(\varphi_1, \varphi_2) = \mathcal{E}\{\langle s, \varphi_1 \rangle \cdot \overline{\langle s, \varphi_2 \rangle}\}.$$

where $\overline{\langle s, \varphi_2 \rangle} = \langle s, \varphi_2 \rangle$ when s is real-valued. Thanks to the moment generating properties of the Fourier transform, this functional can be calculated from the characteristic form $\mathcal{Z}_s(\varphi)$ as

$$\mathcal{B}_s(\varphi_1, \varphi_2) = (-j)^2 \frac{\partial^2 \mathcal{Z}_s(\omega_1 \varphi_1 + \omega_2 \varphi_2)}{\partial \omega_1 \partial \omega_2} \Big|_{\omega_1=0, \omega_2=0}, \quad (26)$$

where we are implicitly assuming that the required partial derivative of the characteristic functional exists. The autocorrelation of the process is then obtained by making the formal substitution $\varphi_1 = \delta(\cdot - t_1)$ and $\varphi_2 = \delta(\cdot - t_2)$:

$$R_s(t_1, t_2) = \mathcal{E}\{s(t_1)s(t_2)\} = \mathcal{B}_s(\delta(\cdot - t_1), \delta(\cdot - t_2)).$$

Alternatively, we can also retrieve the autocorrelation function by invoking the kernel theorem: $\mathcal{B}_s(\varphi_1, \varphi_2) = \int_{\mathbb{R}^2} R_s(t_1, t_2) \varphi_1(t_1) \varphi_2(t_2) dt_1 dt_2$.

The concept also generalizes for the calculation of the higher-order correlation form⁴

$$\mathcal{E}\{\langle s, \varphi_1 \rangle \cdot \langle s, \varphi_2 \rangle \cdots \langle s, \varphi_N \rangle\} = (-j)^N \frac{\partial^N \mathcal{Z}_s(\omega_1 \varphi_1 + \cdots + \omega_N \varphi_N)}{\partial \omega_1 \cdots \partial \omega_N} \Big|_{\omega_1=0, \dots, \omega_N=0}$$

which provides the basis for the determination of higher-order moments and cumulants.

Here, we concentrate on the calculation of the second-order moments, which happen to be independent upon the specific type of noise. For the cases where the covariance is defined and finite, it is not hard to show that the generic covariance form of the white noise processes defined in Section II-C is

$$\mathcal{B}_w(\varphi_1, \varphi_2) = \sigma_0^2 \langle \varphi_1, \varphi_2 \rangle,$$

where σ_0^2 is a suitable normalization constant that depends on the noise parameters (b_1, b_2, V) . We then perform the usual adjoint manipulation to transfer the above formula to the filtered version $s = L^{-1}w$ of such a noise process.

Property 4 (Generalized correlation): The covariance form of the generalized stochastic process whose characteristic form is $\mathcal{Z}_s(\varphi) = \mathcal{Z}_w(L^{-1*}\varphi)$ where \mathcal{Z}_w is a white noise functional is given by

$$\mathcal{B}_s(\varphi_1, \varphi_2) = \sigma_0^2 \langle L^{-1*}\varphi_1, \overline{L^{-1*}\varphi_2} \rangle = \sigma_0^2 \langle \overline{L^{-1}}L^{-1*}\varphi_1, \varphi_2 \rangle,$$

and corresponds to the correlation function

$$R_s(t_1, t_2) = \mathcal{E}\{s(t_1) \cdot \overline{s(t_2)}\} = \sigma_0^2 \langle \overline{L^{-1}}L^{-1*}\delta(\cdot - t_1), \delta(\cdot - t_2) \rangle.$$

The latter characterization requires the determination of the impulse response of $\overline{L^{-1}}L^{-1*}$. In particular, when L^{-1} is LSI with convolution kernel $\rho_L \in L_1$, we get that

$$R_s(t_1, t_2) = \sigma_0^2 \overline{L^{-1}}L^{-1*}\delta(t_2 - t_1) = r_s(t_2 - t_1) = \sigma_0^2 (\bar{\rho}_L * \rho_L^\vee)(t_2 - t_1),$$

⁴For simplicity, we are only giving the formula for a real-valued process.

which confirms that the underlying process is wide-sense stationary. Since the autocorrelation function $r_s(\tau)$ is integrable, we also have a one-to-one correspondence with the traditional notion of power spectrum: $\Phi_s(\omega) = \mathcal{F}\{r_s\}(\omega) = \frac{\sigma_0^2}{|\hat{L}(-\omega)|^2}$, where $\hat{L}(\omega)$ is the frequency response of the whitening operator L .

The determination of the correlation function for the non-stationary processes associated with the unstable versions of (15) is more involved. It can be bypassed if, instead of $s(t)$, we consider the generalized increment process $s_d(t) = \Delta_\alpha s(t)$ of Property 3.

Property 5 (Reduction of correlation distances): Let s be a generalized stochastic process whose characteristic form is $\mathcal{Z}_s(\varphi) = \mathcal{Z}_w(L^{-1*}\varphi)$ where \mathcal{Z}_w is a white noise functional and where L^{-1*} is given by (20) (differential system of order N with pole vector α and driving operator $Q_M(D) = \sum_{m=1}^M b_m D^m$). Then, the correlation form of $s_d(t) = \Delta_\alpha s(t)$ can be written as

$$\mathcal{B}_{s_d}(\varphi_1, \varphi_2) = \sigma_0^2 \langle \beta_L^\vee * \varphi_1, \overline{\beta_L}^\vee * \varphi_2 \rangle,$$

where β_L is the generalized B-spline defined by (25). The corresponding covariance function is

$$R_{s_d}(t_1, t_2) = \mathcal{E} \left\{ \Delta_\alpha s(t_1) \cdot \overline{\Delta_\alpha s(t_2)} \right\} = \sigma_0^2 (\overline{\beta_L} * \beta_L^\vee)(t_2 - t_1)$$

which vanishes for $(t_2 - t_1) \notin [-N, N]$.

The above result is universal in the sense that it does not distinguish between the stable and unstable cases; it can handle N th-order systems in full generality.

The conclusion of this section is that one can apply standard techniques from system theory (determination of impulse responses) to determine the characteristic form of the whole class of Gaussian and non-Gaussian stationary processes defined by (15). The functional tools from spline theory (exponential B-spline calculus) are helpful as well, especially for the handling and characterization of the non-stationary variants of these stochastic processes.

V. LÉVY PROCESSES REVISITED

We now illustrate our method by specifying classical Lévy processes—denoted by $W(t)$ —via the solution of the (marginally unstable) stochastic differential equation

$$\frac{d}{dt}W(t) = w(t) \tag{27}$$

where the driving term w is one of the independent noise processes defined earlier. We shall consider the solution $W(t)$ for all $t \in \mathbb{R}$, but we shall impose the boundary condition $W(t_0) = 0$ with $t_0 = 0$ to make our construction compatible with the classical one which is defined for $t \geq 0$.

In writing Eq. (27), we have intentionally abused the notation in order to be provocative: indeed, when w is Gaussian, the solution of this differential equation turns out to be the classical Wiener process, which is known to be continuous everywhere (almost surely), but nowhere differentiable in the classical sense! Of course, this only makes sense if we interpret (27) as $\langle DW, \varphi \rangle = \langle w, \varphi \rangle$ for all $\varphi \in \mathcal{S}$.

A. Distributional characterization of Lévy processes

The direct application of the operator formalism developed in Section III yields the solution of (27):

$$W(t) = I_{0,0}w(t) = \begin{cases} \int_0^t w(t) dt, & t \geq 0 \\ -\int_t^0 w(t) dt, & t < 0 \end{cases}$$

where $I_{0,0}$ is the unique right inverse of D that imposes the required boundary condition at $t = 0$. The Fourier-based expression of this anti-derivative operator is obtained from the 6th line of Table I by setting $(\omega_0, t_0) = (0, 0)$

$$I_{0,0}s(t) = \int_{\mathbb{R}} \hat{s}(\omega) \frac{e^{j\omega t} - 1}{j\omega} \frac{d\omega}{2\pi},$$

while its adjoint is specified by

$$I_{0,0}^*\varphi(t) = \int_{\mathbb{R}} \frac{\hat{\varphi}(\omega) - \hat{\varphi}(0)}{-j\omega} e^{j\omega t} \frac{d\omega}{2\pi}. \quad (28)$$

We can then invoke Property 2 to obtain the characteristic form of the Lévy process

$$\mathcal{Z}_W(\varphi) = \mathcal{Z}_w(I_{0,0}^*\varphi) \quad (29)$$

which is admissible provided that the Lévy function f fulfils the condition in Theorem 3.

We get the characteristic function of the sample values of the Lévy process $W(t_1) = \langle W, \delta(\cdot - t_1) \rangle$, by making the substitution $\varphi = \omega_1 \delta(\cdot - t_1)$ in (29): $\mathcal{Z}_W(\omega_1 \delta(\cdot - t_1)) = \mathcal{Z}_w(\omega_1 I_{0,0}^* \delta(\cdot - t_1))$ with $t_1 > 0$. Here, we use the Fourier-domain definition (28) to evaluate $I_{0,0}^* \delta(t - t_1) = 1_{[0, t_1)}(t)$ which happens to be an indicator function. Since $1_{[0, t_1)}(t)$ is equal to one for $t \in [0, t_1)$ and zero elsewhere, it is easy to evaluate the integral over t in (2) where $f(u)$ is given by (3), which yields

$$\mathcal{E}\{e^{j\omega_1 W(t_1)}\} = \exp\left(\int_{\mathbb{R}} f[\omega_1 1_{[0, t_1)}(t)] dt\right) = \exp(t_1 f[\omega_1])$$

This result is equivalent to the celebrated Lévy-Khinchine representation of the process [23], [29], [41].

B. Lévy increment process

The Lévy increment process is defined as $\Delta W(t) = W(t) - W(t-1)$ where the classical finite-difference operator $\Delta = \Delta_{(0)}$ is the discrete counterpart of the whitening operator D . The application of Property 3 yields the characteristic form $\mathcal{Z}_w(\beta_{(0)}^\vee * \varphi)$ where $\beta_{(0)}^\vee(t) = \beta_{(0)}(-t)$ is the anti-causal B-spline of degree 0, which is piecewise-constant and compactly supported in $t \in (-1, 0]$. We get the pointwise characterization in essentially the same way as before: $\mathcal{E}\{e^{j\omega_1 \Delta W(t_1)}\} = \mathcal{Z}_{\Delta W}(\omega_1 \delta(\cdot - t_1)) = \mathcal{Z}_w(\omega_1 \beta_{(0)}^\vee(\cdot - t_1))$. Here too, we can evaluate the integral $\int_{\mathbb{R}} f[\omega_1 \beta_{(0)}^\vee(t - t_1)] dt$ based on the fact that the B-spline $\beta_{(0)}^\vee(t - t_1)$ is equal to one for $t \in (-1 + t_1, t_1]$ and zero elsewhere, which leads to the particularly simple result

$$\mathcal{E}\{e^{j\omega_1 \Delta W(t_1)}\} = \exp(f(\omega_1)) = \hat{p}_{\text{id}}(\omega_1), \quad (30)$$

where $f(u)$ is given by (3). Interestingly, this corresponds to the generic form of the characteristic function of a distribution (p_{id}) that is infinitely-divisible.

To investigate the joint dependencies of the increment process, we turn our attention to the 2-D characteristic function of the joint distribution $p(\Delta W(t_1), \Delta W(t_2))$ which is obtained by evaluating the characteristic form of ΔW for $\varphi = \omega_1 \delta(\cdot - t_1) + \omega_2 \delta(\cdot - t_2)$: $\hat{p}(\omega_1, \omega_2) = \mathcal{Z}_w(\omega_1 \beta_{(0)}^\vee(\cdot - t_1) + \omega_2 \beta_{(0)}^\vee(\cdot - t_2))$. When the B-splines are non-overlapping, that is, when $|t_1 - t_2| \geq 1$, we take advantage of the factorization property of the noise functional to show that

$$\mathcal{E}\{e^{j\omega_1 \Delta W(t_1) + j\omega_2 \Delta W(t_2)}\} = \hat{p}(\omega_1, \omega_2) = \hat{p}_{\text{id}}(\omega_1) \cdot \hat{p}_{\text{id}}(\omega_2)$$

where $\hat{p}_{\text{id}}(\omega_1)$ is defined in (30). This last factorization result implies that the samples of the increment process are independent past the correlation distance of 1 (support of the B-spline), which corresponds to the step size of the finite-difference operator. In particular, this implies that the integer samples of ΔW are i.i.d. random variables; they correspond to a stationary independent discrete noise process whose probability density function $p_{\text{id}}(\Delta W) = \mathcal{F}^{-1}(\hat{p}_{\text{id}}(-\omega))(\Delta W)$ may be obtained by taking the inverse Fourier transform of (30).

Note, however, that the continuous-domain version of the increment process is not rigorously independent: it will exhibit dependencies over a range corresponding to the support of the B-spline. If the driving noise is white with a unit variance, then the autocorrelation function of the increment process is given by

$$\mathcal{E}\{\Delta W(t_1) \cdot \Delta W(t_2)\} = (\beta_{(0)}^\vee * \beta_{(0)})(t_1 - t_2) = \text{tri}(t_1 - t_2) = \beta_{(0,0)}(t_1 - t_2 + 1).$$

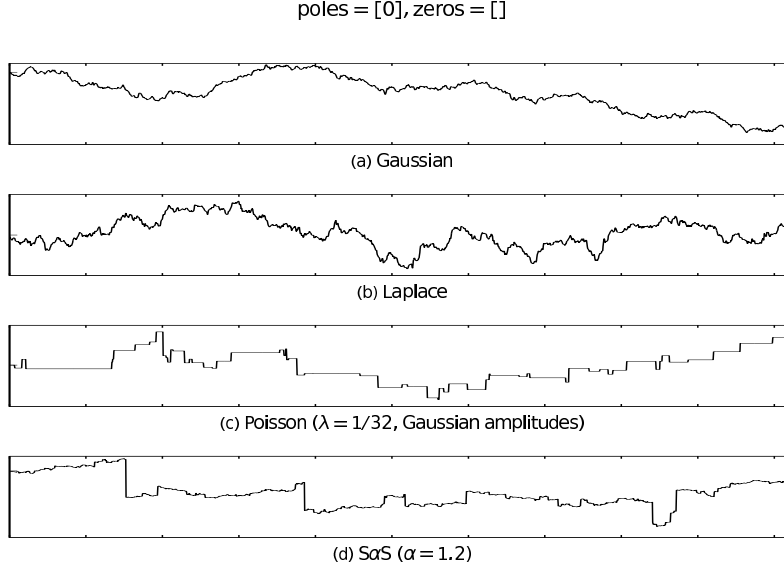


Fig. 1. Examples of Lévy motions $W(t)$ with increasing degrees of sparsity. (a) Brownian motion with Lévy triplet $(0, 1, 0)$. (b) Lévy-Laplace motion with $(0, 0, \frac{e^{-|a|}}{|a|})$. (c) Compound Poisson process with $(0, 0, \lambda \frac{1}{\sqrt{2\pi}} e^{-a^2/2})$ with $\lambda = \frac{1}{32}$. (d) Symmetric Lévy flight with $(0, 0, 1/|a|^{\alpha+1})$ and $\alpha = 1.2$.

C. Examples of Lévy processes

Realizations of four different Lévy processes are shown in Fig. 1 together with their Lévy triplets $(b_1, b_2, v(a))$ where $v(a)$ is the Lévy density such that $v(a)da = V(da)$. The first signal is a Brownian motion (a.k.a. Wiener process) that is obtained by integration of a Gaussian white noise. While the sampled version of ΔW is i.i.d. in all cases, it does not yield a sparse representation in this first instance because the underlying distribution remains Gaussian. The second process, which may be termed Lévy-Laplace motion, is specified by the Lévy density $v(a) = e^{-|a|}/|a|$ which is not in L_1 . By taking the inverse Fourier transform of (30), we can show that its increment process has a Laplace distribution [24]; note that this type of generalized Gaussian model is often used to justify sparsity-promoting signal processing techniques based on ℓ_1 minimization [42]–[44]. The third piecewise-constant signal is a compound Poisson process. It is intrinsically sparse since a good proportion of its increments is zero by construction (with probability $e^{-\lambda}$). The fourth example is an alpha-stable Lévy motion (a.k.a. Lévy flight) with $\alpha = 1.2$. Here, the distribution of ΔW is heavy-tailed (SαS) with unbounded moments for $p > \alpha$. Although this may not be obvious from the picture, this is the sparsest process of the lot because it is ℓ_α -compressible in the strongest sense [37]. Specifically, we can compress the sequence such as to preserve any prescribed portion $r < 1$ of its average ℓ_α energy by retaining an arbitrarily small fraction of samples as the length

of the signal goes to infinity.

D. Link with conventional stochastic calculus

Thanks to (27), we can view a white noise $w = \dot{W}$ as the weak derivative of some classical Lévy processes $W(t)$ which is well-defined pointwise (almost everywhere). This provides us with further insights on the range of admissible white noise processes of Section II.C which constitute the driving terms of the general stochastic differential equation (8). This fundamental observation also makes the connection with stochastic calculus⁵ [26], [45], which avoids the notion of white noise by relying on the use of stochastic integrals of the form

$$s(t) = \int_{\mathbb{R}} \rho(t, t') \, dW(t')$$

where W is a random (signed) measure associated to some canonical Brownian motion (or, by extension, a Lévy process) and where $\rho(t, t')$ is an integration kernel that formally corresponds to our inverse operator L^{-1} .

VI. CONCLUSION

We have set the foundations of a unifying framework that gives access to the broadest possible class of continuous-time stochastic processes that are specifiable by linear, shift-invariant equations, which is beneficial for signal processing purposes. To illustrate the method, we have investigated the extended family of Lévy processes, which, in our view, provide the simplest and most basic examples of sparse processes, despite the fact that they are non-stationary (due to their pole at the origin).

The proposed class of stochastic models and its corresponding mathematical machinery (Fourier analysis, characteristic functional, and exponential B-spline calculus) has a large range of applicability, as we shall illustrate in the companion paper [30]. Our formulation has the advantage of maintaining backward compatibility with the classical theory of Gaussian stationary processes, while introducing a large variety of new processes whose properties are better matched to the currently-dominant paradigm in the field which is focused on the notion of sparsity.

⁵The Itô integral of conventional stochastic calculus is based on Brownian motion, but the concept can also be generalized to Lévy driving terms using the more advanced theory of semimartingales [45].

APPENDIX I: POSITIVE-DEFINITE FUNCTIONALS

We start by recalling the fundamental notion of positive-definiteness for univariate functions [46]. A complex-valued function f of the real variable ω is said to be *positive-definite* iff.

$$\sum_{m=1}^N \sum_{n=1}^N f(\omega_m - \omega_n) \xi_m \bar{\xi}_n \geq 0$$

for every possible choice of $\omega_1, \dots, \omega_N \in \mathbb{R}$, $\xi_1, \dots, \xi_N \in \mathbb{C}$ and $N \in \mathbb{Z}_+$. This is equivalent to the requirement that the $N \times N$ matrix \mathbf{F} whose elements are given by $[\mathbf{F}]_{mn} = f(\omega_m - \omega_n)$ is positive semi-definite (that is, non-negative definite) for all N , no matter how the ω_n 's are chosen.

Bochner's theorem states that a bounded, continuous function f is positive-definite if and only if it is the Fourier transform of a positive and finite Borel measure μ :

$$f(\omega) = \int_{\mathbb{R}} e^{j\omega x} \mu(dx).$$

In particular, Bochner's theorem implies that f is the characteristic function of a random variable—that is, $f(\omega) = \mathcal{E}\{e^{j\omega x}\} = \int_{\mathbb{R}} e^{j\omega x} \mu(dx)$ where x is the random variable with probability measure μ —iff. f is continuous, positive-definite and $f(0) = 1$. Note that the above results and formulas also generalize to the multivariate setting.

These concepts carry over as well to functionals on some abstract nuclear space \mathcal{E} , the prime example being Schwartz's class \mathcal{S} of smooth and rapidly-decreasing test functions [19].

Definition 1: A complex-valued functional $L(\varphi)$ defined over the function space \mathcal{E} is said to be *positive-definite* iff.

$$\sum_{m=1}^N \sum_{n=1}^N L(\varphi_m - \varphi_n) \xi_m \xi_n^* \geq 0$$

for every possible choice of $\varphi_1, \dots, \varphi_N \in \mathcal{E}$, $\xi_1, \dots, \xi_N \in \mathbb{C}$ and $N \in \mathbb{N}^+$.

Theorem 4 (Minlos-Bochner): Given a functional $Z(\varphi)$ on a nuclear space \mathcal{E} that is continuous, positive-definite and such that $Z(0) = 1$, there exists a unique probability measure μ on the dual space \mathcal{E}' such that

$$Z(\varphi) = \mathcal{E}\{e^{j\langle s, \varphi \rangle}\} = \int_{\mathcal{E}'} e^{j\langle s, \varphi \rangle} d\mu(s),$$

where $\langle s, \varphi \rangle$ is the dual pairing map. One further has the guarantee that all finite dimensional probabilities densities that can be derived from $Z(\varphi)$ by setting $\varphi = \omega_1 \varphi_1 + \dots + \omega_N \varphi_N$ are mutually compatible.

The characteristic form therefore uniquely specifies the generalized stochastic process $s = s(\varphi)$ in essentially the same way that the characteristic function fully determines the probability measure of a scalar or multivariate random variable.

APPENDIX II: PROOF OF THEOREM 3

1) As w is a generalized random process, \mathcal{Z}_w is a continuous functional on \mathcal{S} . This, together with the assumption that T is a continuous operator on \mathcal{S} , implies that the composed functional $\mathcal{Z}_s(\varphi) := \mathcal{Z}_w(T\varphi)$ is continuous on \mathcal{S} .

Given the functions $\varphi_1, \dots, \varphi_N$ in \mathcal{S} and some complex coefficients ξ_1, \dots, ξ_N ,

$$\begin{aligned}
& \sum_{1 \leq m, n \leq N} \mathcal{Z}_s(\varphi_m - \varphi_n) \xi_m \overline{\xi_n} \\
&= \sum_{1 \leq m, n \leq N} \mathcal{Z}_w(T(\varphi_m - \varphi_n)) \xi_m \overline{\xi_n} \\
&= \sum_{1 \leq m, n \leq N} \mathcal{Z}_w(T\varphi_m - T\varphi_n) \xi_m \overline{\xi_n} \quad (\text{by the linearity of the operator } T) \\
&\geq 0. \quad (\text{by the positivity of } \mathcal{Z}_w)
\end{aligned}$$

This proves the positive-definiteness of the functional \mathcal{Z}_s on \mathcal{S} .

Clearly, $\mathcal{Z}_s(0) = \mathcal{Z}_w(T0) = \mathcal{Z}_w(0) = 1$.

2) By the assumption on the operator T , $T\varphi \in L_p$ for all $\varphi \in \mathcal{S}$. This together with the assumption $|f(u)| \leq C|u|^p$ implies that $\mathcal{Z}_s(\varphi) = \exp(\int_{\mathbb{R}} f(T\varphi(t)) dt)$ is well-defined for all $\varphi \in \mathcal{S}$. By the linear property of the operator T and $f(0) = 0$, we obtain that $\mathcal{Z}_s(0) = 1$. The positive-definiteness of the functional \mathcal{Z}_s is established by an argument similar to the one used above. Finally we prove the continuity of the functional \mathcal{Z}_s on \mathcal{S} : Let $\{\varphi_n\}_{n=1}^{\infty}$ be a convergent sequence in \mathcal{S} and denote its limit in \mathcal{S} by φ . Then by the assumption on the linear operator T , $T\varphi_n$ converges to $T\varphi$ in L_p ; that is,

$$\lim_{n \rightarrow \infty} \|T\varphi_n - T\varphi\|_p = 0. \quad (31)$$

Next, we observe that

$$\begin{aligned}
|f(u) - f(v)| &= \left| \int_v^u f'(t) dt \right| \\
&\leq C \left| \int_v^u t^{p-1} dt \right| \quad (\text{by the assumption on } f) \\
&\leq C \max(|u|^{p-1}, |v|^{p-1}) |u - v| \\
&\leq C(|v|^{p-1} + |u - v|^{p-1}) |u - v|. \quad (\text{by the triangle inequality})
\end{aligned}$$

We then have

$$\begin{aligned}
& \left| \int_{\mathbb{R}} f(T\varphi_n(t)) \, dt - \int_{\mathbb{R}} f(T\varphi(t)) \, dt \right| \\
& \leq C \int_{\mathbb{R}} |T\varphi(t)|^{p-1} |T\varphi_n(t) - T\varphi(t)| + |T\varphi_n(t) - T\varphi(t)|^p \, dt \\
& \leq C \left(\|T\varphi\|_p^{p-1} \|T\varphi_n - T\varphi\|_p + \|T\varphi_n - T\varphi\|_p^p \right) \quad (\text{by Hölder's inequality}) \\
& \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (\text{by (31)})
\end{aligned}$$

which proves the continuity of the functional \mathcal{Z}_s on \mathcal{S} . ■

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